A GENERALIZED JACOBI DISTRIBUTION

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Abstract

In this paper a new generalized Jacobi distribution, which involves ${}_p\Psi_q-$function is defined. Some statistical functions associated with the probability density function, such as, the characteristic function, the survivor function, the cumulative density function, basic moments and the hazard rate function, have been deduced. Further some figures are drawn for the probability mass function, the cumulative density function and the hazard rate function, to show the effect of the parameters involved. Results given by Ben Nakhi and Kalla and Sarabia and Kalla can be derived from our formulas.

Keywords: Generalized Jacobi distribution; Probability density function; Characteristic function; Survivor function; Cumulative density function; Moments; Hazard rate function.

1 Introduction

A fairly wide range of special functions can be represented in terms of the hypergeometric and confluent hypergeometric series. Hypergeometric series in one and more variables occur naturally in a wide variety of problems in applied mathematics, statistic, operations research, theoretical physics, and engineering science ([2],[7],[9],[13],[17]).

Kalla et al. [8] have used hypergeometric functions to study a unified form of gamma-type distributions; Al-Saqabi and Kalla [1] have considered a probability distribution involving a confluent hypergeometric function of two variables; Ben Nakhi and Kalla [3,5] established several mixture distributions which are obtained by mixing discrete distributions with continuous ones, whereas a new mixture distribution associated with Fox-Wright generalized hypergeometric function has been studied by Saxena and Kalla [12].

Statistical distributions have been used in a variety of applications, including the field of reliability, accident proneness and entomological field data [6].

Sarabia and Kalla [11] have studied the following probability density function (pdf):

$$h(x) = \frac{(1-x)^a(1+x)^b}{2^{a+b+1}\ B(a+1,b+1)\ R}\ {}_3F_2\left(\begin{matrix}-\nu, \nu+\lambda, c\\ \alpha+1, p\end{matrix}\ ; \tfrac{1-x}{2}\right), \quad -1 \le x \le 1 \tag{1}$$

with

$$R = {}_4F_3\left(\begin{matrix}-\nu, \nu+\lambda, c, a+1\\ \alpha+1, p, a+b+2\end{matrix}\ ; 1\right).$$

On the other hand, Ben Nakhi and Kalla [4] introduce a $w-$Jacobi random variable whose pdf is given by

$$g(x) = g^{a,b,c,p;\nu}_{w;\alpha,\beta}(x)$$

$$= \frac{(1-x)^a(1+x)^b}{2^{a+b+1}\ B(a+1,b+1)\ \overset{w}{R}}\ {}_3\overset{w}{R}_2\left(\begin{matrix}-\nu, \underline{\nu+\lambda}, c\\ \underline{\alpha+1}, p\end{matrix}\ ; \tfrac{1-x}{2}\right), -1 \le x \le 1 \tag{2}$$

where

$$\overset{w}{R} = {}_4\overset{w}{R}_3\left(\begin{array}{c} -\nu, \underline{\nu+\lambda}, c, a+1 \\ \underline{\alpha+1}, p, a+b+2 \end{array}; 1\right)$$

and ${}_3\overset{w}{R}_2(.)$ is the generalized $w-$Gauss hypergeometric function defined by

$${}_3\overset{w}{R}_2\left(\begin{array}{l} a_1, \underline{a_2}, a_3 \\ \underline{b_1}, b_2 \end{array}; y\right) = \frac{\Gamma(b_1)}{\Gamma(a_2)} \sum_{k=0}^{\infty} \frac{\Gamma(a_2+wk)(a_1)_k(a_3)_k}{\Gamma(b_1+wk)(b_2)_k} \frac{y^k}{k!}.$$

In this paper a new generalized Jacobi distribution, which involves ${}_p\Psi_q$ $-$ function is defined. Some statistical functions associated with the probability density function, such as, the characteristic function, the survivor function, the cumulative density function, basic moments and the hazard rate function, have been deduced. Further some figures are drawn for the probability mass function, the cumulative density function and the hazard rate function, to show the effect of the parameters involved. Results given by Ben Nakhi and Kalla [4] and Sarabia and Kalla [11] can be derived from our formulas.

2 Preliminaries

In this section some special functions neccesary for the development of other sections are presented.

The Kampé de Fériet function has been generalized by Srivastava and Daoust ([14],[16]). Their general function is defined and represented as follows:

$$S^{p_1:p_2;p_3}_{q_1:q_2;q_3}\left[\begin{array}{lll} (a_j;\alpha_j,A_j)_{1,p_1}: & (c_j,\gamma_j)_{1,p_2}; & (e_j,E_j)_{1,p_3}; \\ (b_j;\beta_j,B_j)_{1,q_1}: & (d_j,\delta_j)_{1,q_2}; & (f_j,F_j)_{1,q_3}; \end{array} x, y\right] =$$

$$\sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p_1}\Gamma\left(a_j+\alpha_j m+A_j n\right)\prod_{j=1}^{p_2}\Gamma\left(c_j+\gamma_j m\right)\prod_{j=1}^{p_3}\Gamma\left(e_j+E_j n\right)}{\prod_{j=1}^{q_1}\Gamma\left(b_j+\beta_j m+B_j n\right)\prod_{j=1}^{q_2}\Gamma\left(d_j+\delta_j m\right)\prod_{j=1}^{q_3}\Gamma\left(f_j+F_j n\right)} \frac{x^m}{m!}\frac{y^n}{n!} \tag{3}$$

where $(a_j;\alpha_j,A_j)_{1,p_1}$ abbreviates the array of p_1 parameters $(a_1;\alpha_1,A_1), \ldots..,$ $(a_{p_1};\alpha_{p_1},A_{p_1})$, and so on.

The series given by (3) converges absolutely, if

$$1+\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{q_2}\delta_j-\sum_{j=1}^{p_1}\alpha_j-\sum_{j=1}^{p_2}\gamma_j\geq 0$$

and

$$1+\sum_{j=1}^{q_1}B_j+\sum_{j=1}^{q_3}F_j-\sum_{j=1}^{p_1}A_j-\sum_{j=1}^{p_3}E_j\geq 0$$

where each of the equalities holds when the variables are suitably constrained [15].

An interesting generalization of the series ${}_pF_q(.)$ is due to Fox and Wright [17] who studied the asymptotic expansion of the generalized hypergeometric function defined by

$$ {}_p\Psi_q\left[\begin{array}{l}(a_j,A_j)_{1,p};\\ (b_j,B_j)_{1,q};\end{array} x\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(a_j+A_jn\right)}{\prod_{j=1}^{q}\Gamma\left(b_j+B_jn\right)}\frac{x^n}{n!} \tag{4}$$

where the coefficients $A_1, ..., A_p$ and $B_1, ..., B_q$ are positive real numbers such that

$$1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\geq 0.$$

From (4) and using the definition of the beta-function we can easily establish the following integral:

$$\begin{aligned}&\int_{-1}^{1}(1-x)^a(1+x)^b{}_p\Psi_q\left[\begin{array}{l}(a_j,A_j)_{1,p};\\ (b_j,B_j)_{1,q};\end{array}\frac{1-x}{2}\right]dx\\ =\ &2^{a+b+1}\Gamma(b+1)\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{l}(a+1,1),(a_j,A_j)_{1,p};\\ (a+b+2,1),(b_j,B_j)_{1,q};\end{array} 1\right]\end{aligned} \tag{5}$$

$$\mathrm{Re}(a),\mathrm{Re}(b)>-1,\ 1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\geq 0.$$

3 Generalized Jacobi distribution

We say that X has a generalized Jacobi distribution with parameters a, b, a_i, A_i $(i = 1, 2, ..., p)$, b_j, B_j $(j = 1, 2, ..., q)$ if X has a probability mass function (pmf)

$$\begin{aligned} f(x) &= f^{a,b;a_1,...,a_p;b_1,...,b_q}_{A_1,...,A_p;B_1,...,B_q}(x) \\ &= \frac{(1-x)^a(1+x)^b}{2^{a+b+1}\ \Gamma(b+1)} \frac{{}_p\Psi_q\left[\begin{array}{l}(a_j,A_j)_{1,p};\\ (b_j,B_j)_{1,q};\end{array} \frac{1-x}{2}\right]}{{}_{p+1}\Psi_{q+1}\left[\begin{array}{l}(a+1,1),(a_j,A_j)_{1,p};\\ (a+b+2,1),(b_j,B_j)_{1,q};\end{array} 1\right]}, \end{aligned} \tag{6}$$

$$\begin{gathered} \mathrm{Re}(a), \mathrm{Re}(b) > 0, a_i, A_i\ (i = 1, 2, ..., p),\ b_j, B_j\ (j = 1, 2, ..., q) > 0, \\ -1 \leq x \leq 1,\ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \end{gathered} \tag{7}$$

3.1 Characteristic function

The characteristic function of X, for any real t, is given by

$$\begin{aligned} \varphi_{\mathrm{X}}(t) &\triangleq E\left[e^{itX}\right] = E\left[e^{it(1-(1-X))}\right] \\ &= e^{it}\sum_{n=0}^{\infty} \frac{(-it)^n}{n!} E\left[(1-X)^n\right]. \end{aligned} \tag{8}$$

By definition,

$$E[(1-X)^n] = \int_{-1}^{1} (1-t)^n f(t)dt$$

from (6),

$$E[(1-X)^n] = \frac{1}{2^{a+b+1}\ \Gamma(b+1)\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{l}(a+1,1),(a_j,A_j)_{1,p};\\ (a+b+2,1),(b_j,B_j)_{1,q};\end{array} 1\right]}\ \mathrm{x}$$

$$\int_{-1}^{1}(1-t)^{a+n}(1+t)^b\ {}_p\Psi_q\left[\begin{array}{l}(a_j,A_j)_{1,p};\\ (b_j,B_j)_{1,q};\end{array} \frac{1-t}{2}\right] dt$$

and by virtue of the result (5)

$$E[(1-X)^n] = \frac{2^n \;{}_{p+1}\Psi_{q+1}\left[\begin{array}{l}(a+n+1,1),(a_j,A_j)_{1,p};\\(a+n+b+2,1),(b_j,B_j)_{1,q};\end{array}\ 1\right]}{{}_{p+1}\Psi_{q+1}\left[\begin{array}{l}(a+1,1),(a_j,A_j)_{1,p};\\(a+b+2,1),(b_j,B_j)_{1,q};\end{array}\ 1\right]}. \tag{9}$$

Substituting this result in (8) and using (4), we obtain

$$\varphi_{\mathrm{X}}(t) = \frac{e^{it}}{{}_{p+1}\Psi_{q+1}\left[\begin{array}{l}(a+1,1),(a_j,A_j)_{1,p};\\(a+b+2,1),(b_j,B_j)_{1,q};\end{array}\ 1\right]} \quad \mathrm{x}$$

$$\sum_{n,k=0}^{\infty} \frac{\Gamma(a+n+1+k)\prod_{j=1}^{p}\Gamma\left(a_j+A_jk\right)}{\Gamma(a+n+b+2+k)\prod_{j=1}^{q}\Gamma\left(b_j+B_jk\right)} \frac{2^n(-it)^n}{n!k!}$$

finally,

$$\varphi_{\mathrm{X}}(t) \triangleq E\left[e^{itX}\right] = \frac{e^{it}}{{}_{p+1}\Psi_{q+1}\left[\begin{array}{l}(a+1,1),(a_j,A_j)_{1,p};\\(a+b+2,1),(b_j,B_j)_{1,q};\end{array}\ 1\right]} \quad \mathrm{x}$$

$$S^{1:p;0}_{1:q;0}\left[\begin{array}{lll}(a+1;1,1): & (a_j,A_j)_{1,p}; & ----;\\(a+b+2;1,1): & (b_j,B_j)_{1,q}; & ----;\end{array}\ 1,-2it\right] \tag{10}$$

where we have used (3).

3.2 The survivor function and the cumulative density function

The survivor function $S(x)$ of the random variable X is given by

$$S(x) = \frac{\left(\frac{1-x}{2}\right)^{a+1}}{\Gamma(-b)\Gamma(b+1)\;{}_{p+1}\Psi_{q+1}\left[\begin{array}{l}(a+1,1),(a_j,A_j)_{1,p};\\(a+b+2,1),(b_j,B_j)_{1,q};\end{array}\ 1\right]} \quad \mathrm{x}$$

$$S^{1:p;1}_{1:q;0}\left[\begin{array}{lll} (a+1;1,1): & (a_j,A_j)_{1,p}; & (-b,1);\\ (a+2;1,1): & (b_j,B_j)_{1,q}; & ---; \end{array} \frac{1-x}{2},\frac{1-x}{2}\right]. \qquad (11)$$

Proof

By definition

$$\begin{aligned} S(x) &\triangleq P(X\geq x)\\ &= \int_x^1 f(t)dt \end{aligned}$$

and using (6)

$$S(x) = \frac{1}{\Gamma(b+1)_{p+1}\Psi_{q+1}\left[\begin{array}{l}(a+1,1),(a_j,A_j)_{1,p};\\(a+b+2,1),(b_j,B_j)_{1,q};\end{array} 1\right]} \qquad \text{x}$$

$$\int_0^{\frac{1-x}{2}} u^a(1-u)^b\ {}_p\Psi_q\left[\begin{array}{l}(a_j,A_j)_{1,p};\\(b_j,B_j)_{1,q};\end{array} u\right] du$$

where we have made a simple variable change.

From the definition (4) and the results [10, p. 797]

$$B_x(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt,\ \mathrm{Re}(\alpha)>0,\ x<1$$

$$B_x(\alpha,\beta) = \frac{x^\alpha}{\alpha}\ {}_2F_1(\alpha,1-\beta;\alpha+1;x)$$

$$|\arg x|\,,\ |\arg(1-x)|<\pi$$

we obtain,

$$S(x) = \frac{\left(\frac{1-x}{2}\right)^{a+1}}{\Gamma(-b)\Gamma(b+1)\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{l}(a+1,1),(a_j,A_j)_{1,p};\\(a+b+2,1),(b_j,B_j)_{1,q};\end{array} 1\right]} \qquad \text{x}$$

$$\sum_{h,k=0}^{\infty} \frac{\Gamma(a+1+h+k)\Gamma(-b+h)\prod_{j=1}^{p}\Gamma\left(a_j+A_jk\right)}{\Gamma(a+2+h+k)\prod_{j=1}^{q}\Gamma\left(b_j+B_jk\right)\ \ h!k!}\left(\frac{1-x}{2}\right)^{h+k}$$

and applying (3) we have the desired result.

On the other hand, the cumulative density function (c.d.f.) is defined as

$$\begin{aligned} F(x) &\triangleq P(X \leq x) = \int_{-1}^{x} f(t)dt \\ &= \int_{-1}^{1} f(t)dt - \int_{x}^{1} f(t)dt \\ &= 1 - S(x) \end{aligned} \tag{12}$$

where $S(x)$ is given by (11).

3.3 Basic moments

In this subsection we obtain the basic moments, such as, the $r-th$ moment, the mean and the moment generating function.

i) **The r − th moment:** For positive integer r

$$\begin{aligned} E\left[X^r\right] &= E\left[(1-(1-X))^r\right] \\ &= \sum_{n=0}^{r}(-1)^n \binom{r}{n} E\left[(1-X)^n\right] \end{aligned}$$

and from (9) we have

$$E\left[X^r\right] = \frac{1}{{}_{p+1}\Psi_{q+1}\left[\begin{array}{l} (a+1,1),(a_j,A_j)_{1,p}; \\ (a+b+2,1),(b_j,B_j)_{1,q}; \end{array} 1\right]} \qquad \text{x}$$

$$\sum_{n=0}^{r}(-2)^n \left(\begin{array}{c} r \\ n \end{array} \right) {}_{p+1}\Psi_{q+1} \left[\begin{array}{l} (a+n+1,1),(a_j,A_j)_{1,p}; \\ (a+n+b+2,1),(b_j,B_j)_{1,q}; \end{array} 1 \right]. \qquad (13)$$

ii) The mean

Since the mean, expected value of the random variable X, is the first moment,

$$\begin{aligned} E[X] &= \sum_{n=0}^{1}(-1)^n \left(\begin{array}{c} 1 \\ n \end{array} \right) E\left[(1-X)^n\right] \\ &= 1 - E\left[(1-X)\right] \end{aligned}$$

that is,

$$E[X] = 1 - \frac{2 \; {}_{p+1}\Psi_{q+1} \left[\begin{array}{l} (a+2,1),(a_j,A_j)_{1,p}; \\ (a+b+3,1),(b_j,B_j)_{1,q}; \end{array} 1 \right]}{{}_{p+1}\Psi_{q+1} \left[\begin{array}{l} (a+1,1),(a_j,A_j)_{1,p}; \\ (a+b+2,1),(b_j,B_j)_{1,q}; \end{array} 1 \right]} \qquad (14)$$

where we have used (9).

The variance of the random variable X, denoted by σ_{X}^2, can be obtained from (13) and (14), since that

$$\sigma_{\mathrm{X}}^2 \triangleq E\left[X^2\right] - \left(E[X]\right)^2. \qquad (15)$$

iii) Moment generating function

The moment generating function of X is obtained from (10) taking $\tau = it$, so

$$\begin{aligned} M_{\mathrm{X}}(\tau) &\triangleq E\left[e^{\tau X}\right] \\ &= \frac{e^{\tau}}{{}_{p+1}\Psi_{q+1} \left[\begin{array}{l} (a+1,1),(a_j,A_j)_{1,p}; \\ (a+b+2,1),(b_j,B_j)_{1,q}; \end{array} 1 \right]} \qquad \text{x} \end{aligned}$$

$$S^{1:p;0}_{1:q;0}\left[\begin{array}{lll} (a+1;1,1): & (a_j,A_j)_{1,p}; & ----; \\ (a+b+2;1,1): & (b_j,B_j)_{1,q}; & ----; \end{array} 1,-2\tau\right]. \tag{16}$$

3.4 Hazard rate function

The failure rate called hazard rate function is defined by

$$h(x)=\frac{f(x)}{S(x)}. \tag{17}$$

From (6) and (11) we can write

$$h(x)=\frac{\Gamma(-b)(1+x)^b}{2^b\ (1-x)} \qquad \text{x}$$

$$\frac{{}_p\Psi_q\left[\begin{array}{l} (a_j,A_j)_{1,p}; \\ (b_j,B_j)_{1,q}; \end{array} \begin{array}{c}\frac{1-x}{2}\end{array}\right]}{S^{1:p;1}_{1:q;0}\left[\begin{array}{lll} (a+1;1,1): & (a_j,A_j)_{1,p}; & (-b,1); \\ (a+2;1,1): & (b_j,B_j)_{1,q}; & ---; \end{array} \frac{1-x}{2},\frac{1-x}{2}\right]}. \tag{18}$$

The results (10), (11), (13), (14), (16) and (18) are valid under the conditions given in (7).

4 Particular cases

If in the results (6), (9), (10), (11), (13) and (14) we put:
i) $p=3,\ q=2;\ a_1=-\nu,\ a_2=\nu+\lambda,\ a_3=c,\ b_1=\alpha+1,\ b_2=p;\ A_1=1,$ $A_2=w,\ A_3=1, B_1=w,\ B_2=1$ we obtain the results given by Ben Nakhi and Kalla in [3].
ii) $p=3,\ q=2;\ a_1=-\nu,\ a_2=\nu+\lambda,\ a_3=c,\ b_1=\alpha+1,\ b_2=p;\ A_1=A_2$ $=A_3=B_1=B_2=1$ we obtain the results given by Sarabia and Kalla in [11].
Now, we present some figures which show the effect of the parameters a and b.
In all figures we use the following selected values:
$p=3,\quad q=2;$
$a_1=1,\ A_1=2.5;\ \ a_2=2.5,\ A_2=2;\ \ a_3=4,\ A_3=3.5;$
$b_1=0.8,\ B_1=5.5;\ \ b_2=1.2,\ B_2=6;$

Fig. 1, Fig. 2 and Fig. 3 show the probability mass function $f(x)$, the cumulative density function $F(x)$ and the hazard rate function $h(x)$ respectively for $b = 1.5$.
The curves (from left to right) correspond to the following selected values for a: $3,\ 2,\ 1,\ 0.5,\ 0.1$.

Fig. 4, Fig. 5 and Fig. 6 show the probability mass function $f(x)$, the cumulative density function $F(x)$ and the hazard rate function $h(x)$ respectively for $a = 0.5$.
The curves (from left to right) correspond to the following selected values for b: $1,\ 2,\ 3,\ 5,\ 10$.

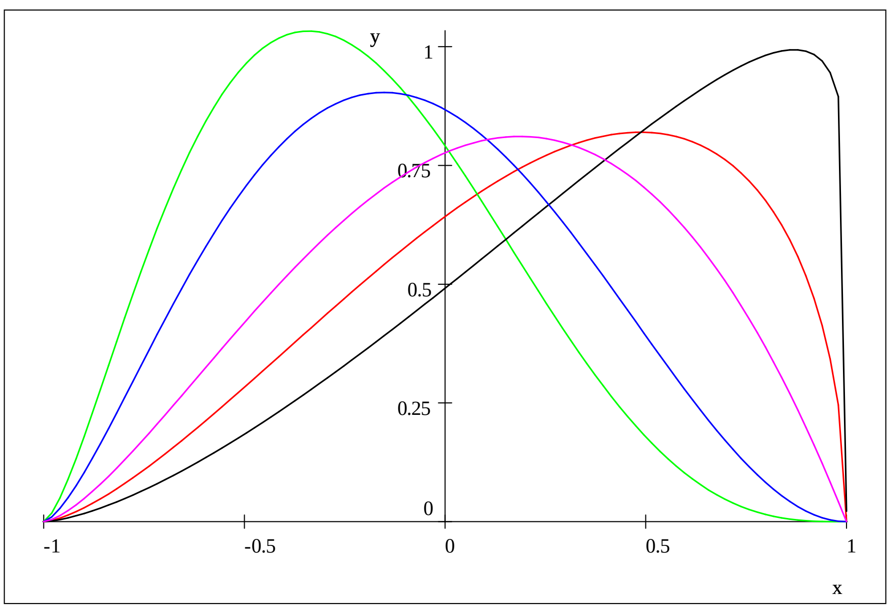

Fig. 1: The probability mass function $f(x)$ for $b = 1.5$ and different values of a.

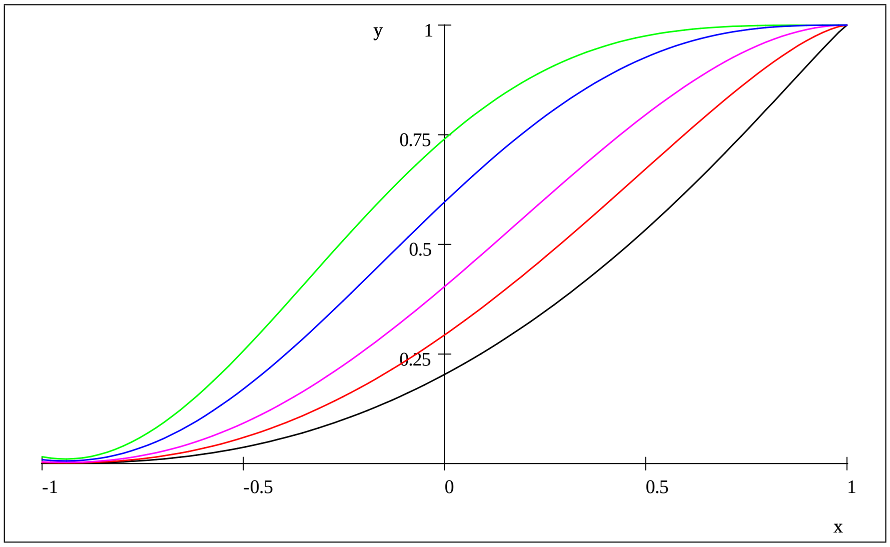

Fig. 2: The cumulative density function $F(x)$ for $b = 1.5$ and different values of a.

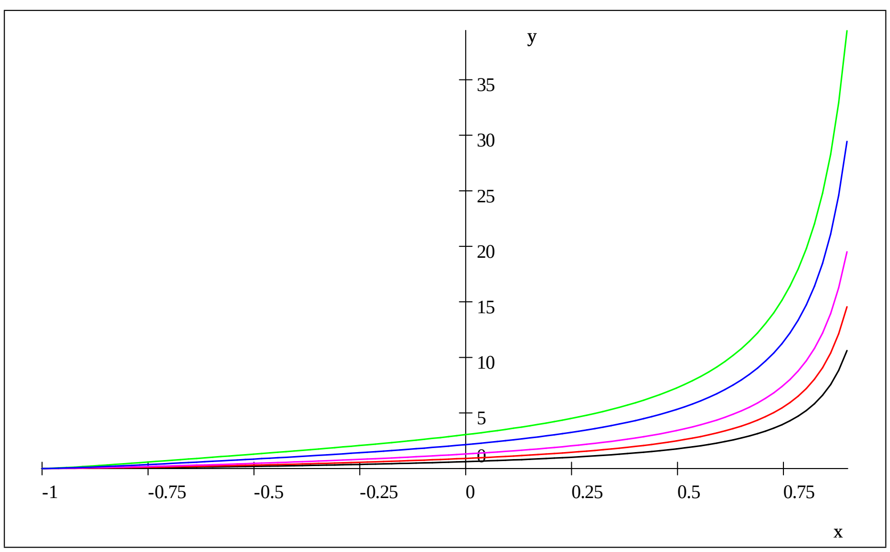

Fig. 3: The hazard rate function $h(x)$ for $b = 1.5$ and different values of a.

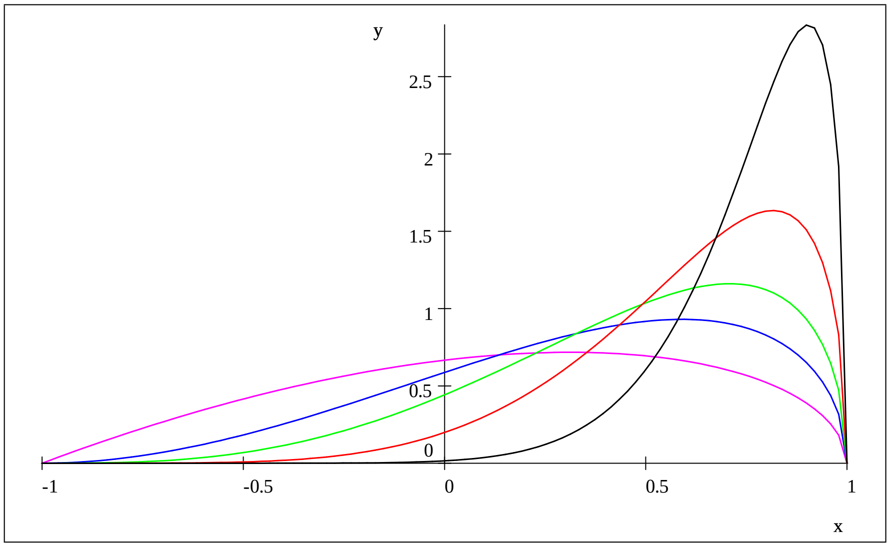

Fig. 4: The probability mass function $f(x)$ for $a = 0.5$ and different values of b.

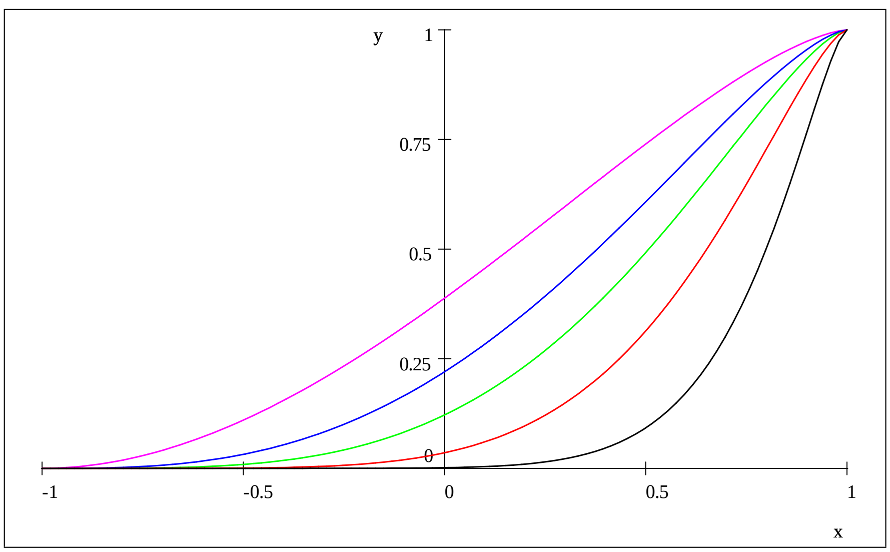

Fig. 5: The cumulative density function $F(x)$ for $a = 0.5$ and different values of b.

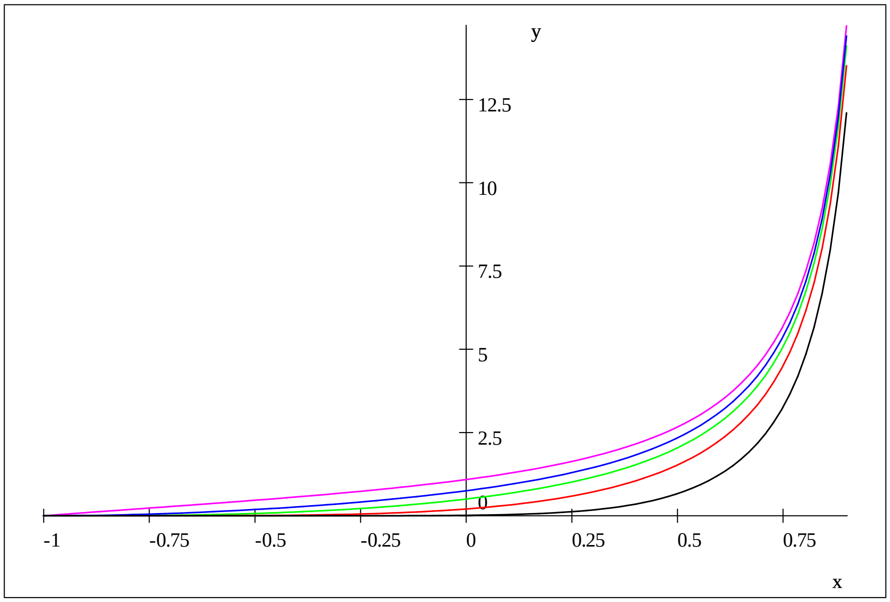

Fig. 6: The hazard rate function $h(x)$ for $a = 0.5$ and different values of b.

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